Week 6 - Monday

COMP 2230

Last time

- Recurrence relations
- Solving recurrence by iteration

Questions?

Assignment 2

Logical warmup

- Here is the tale of the mathematician Diophantus, the father of algebra:
  - Diophantus's youth lasted $\frac{1}{6}$ of his life.
  - He grew his first beard after another $\frac{1}{12}$ of his life.
  - At the end of the following $\frac{1}{7}$ of his life, he married.
  - Five years later, his son was born.
  - His son lived exactly $\frac{1}{2}$ of Diophantus's life.
  - Diophantus died four years after the death of his son.
- How long did Diophantus live?

Solving Second-Order Linear Homogeneous Relations with Constant Coefficients

Second-Order Linear Homogeneous Relations with Constant Coefficients

- Second-order linear homogeneous relations with constant coefficients are recurrence relations of the following form:
  - $a_k = Aa_{k-1} + Ba_{k-2}$ where $A, B \in \mathbb{R}$ and $B \neq 0$
- These relations are:
  - **Second order** because they depend on a_{k-1} and a_{k-2}
  - **Linear** because a_{k-1} and a_{k-2} are to the first power and not multiplied by each other
  - **Homogeneous** because there is no constant term
  - **Constant coefficients** because A and B are fixed values

Why do we care?

- I'm sure you're thinking that this is an awfully narrow class of recurrence relations to have special rules for
- It's true: There are many (infinitely many) ways to formulate a recurrence relation
  - Some have explicit formulas
  - Some do not have closed explicit formulas
- We care about this one partly for two reasons
  1. We can solve it
  2. It lets us get an explicit formula for Fibonacci!

Pick 'em out

- Which of the following are second-order linear homogeneous recurrence relations with constant coefficients?

a) $a_k = 3a_{k-1} + 2a_{k-2}$
b) $b_k = b_{k-1} + b_{k-2} + b_{k-3}$
c) $c_k = \frac{1}{2}c_{k-1} - \frac{3}{7}c_{k-2}$
d) $d_k = d_{k-1}^2 + d_{k-1}d_{k-2}$
e) $e_k = 2e_{k-2}$
f) $f_k = 2f_{k-1} + 1$
g) $g_k = g_{k-1} + g_{k-2}$
h) $h_k = (-1)h_{k-1} + (k-1)h_{k-2}$

Characteristic equation

- We will use a tool to solve a SOLHRRwCC called its **characteristic equation**
- The characteristic equation of $a_k = Aa_{k-1} + Ba_{k-2}$ is:
  - $t^2 - At - B = 0$ where $t \geq 0$
- Note that the sequence $1, t, t^2, t^3, \dots, t^n$ satisfies $a_k = Aa_{k-1} + Ba_{k-2}$ if and only if t satisfies $t^2 - At - B = 0$

Demonstrating the characteristic equation

- We can see that $1, t, t^2, t^3, \ldots, t^n$ satisfies $a_k = Aa_{k-1} + Ba_{k-2}$ if and only if t satisfies $t^2 - At - B = 0$ by substituting in t terms for a_k as follows:
  - $t^k = At^{k-1} + Bt^{k-2}$

- Since $t \geq 0$, we can divide both sides through by t^{k-2}
  - $t^2 = At + B$
  - $t^2 - At - B = 0$

Using the characteristic equation

- Consider $a_k = a_{k-1} + 2a_{k-2}$
- What is its characteristic equation?
  - $t^2 - t - 2 = 0$
- What are its roots?
  - $t = 2$ and $t = -1$
- What are the sequences defined by each value of t?
  - $1, 2, 2^2, 2^3, \ldots, 2^n, \ldots$
  - $1, -1, 1, -1, \ldots, (-1)^n, \ldots$
- Do these sequences satisfy the recurrence relation?

Finding other sequences

- An infinite number of sequences satisfy the recurrence relation, depending on the initial conditions
- If $r_0, r_1, r_2, \ldots$ and $s_0, s_1, s_2, \ldots$ are sequences that satisfy the same SOLHRRwCC, then, for any $C, D \in \mathbb{R}$, the following sequence also satisfies the SOLHRRwCC
  - $a_n = Cr_n + Ds_n$, for integers $n \geq 0$

Finding a sequence with specific initial conditions

- Solve the recurrence relation $a_k = a_{k-1} + 2a_{k-2}$ where $a_0 = 1$ and $a_1 = 8$
- The result from the previous slide says that, for any sequence r_k and s_k that satisfy a_k
  - $a_n = Cr_n + Ds_n$, for integers $n \geq 0$
- We have these sequences that satisfy a_k
  - $1, 2, 2^2, 2^3, \ldots, 2^n, \ldots$
  - $1, -1, 1, -1, \ldots, (-1)^n, \ldots$
- Thus, we have
  - $a_0 = 1 = C2^0 + D(-1)^0 = C + D$
  - $a_1 = 8 = C2^1 + D(-1)^1 = 2C - D$
- Solving for C and D, we get:
  - $C = 3$
  - $D = -2$
- Thus, our final result is $a_n = 3 \cdot 2^n - 2(-1)^n$

Distinct Roots Theorem

- We can generalize this result
- Let sequence $a_0, a_1, a_2, \ldots$ satisfy:
  - $a_k = Aa_{k-1} + Ba_{k-2}$ for integers $k \geq 2$
- If the characteristic equation $t^2 - At - B = 0$ has two distinct roots r and s, then the sequence satisfies the explicit formula:
  - $a_n = Cr^n + Ds^n$
  - where C and D are determined by a_0 and a_1

Now we can solve Fibonacci!

- Fibonacci is defined as follows:
  - $F_k = F_{k-1} + F_{k-2}, k \geq 2$
  - $F_0 = 1$
  - $F_1 = 1$
- What is its characteristic equation?
  - $t^2 - t - 1 = 0$
- What are its roots?
- $t = \frac{1 \pm \sqrt{5}}{2}$
- Thus, $F_n = C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n$

We just need C and D

- So, we plug in $F_0 = 1$ and $F_1 = 1$
- $F_0 = C\left(\frac{1+\sqrt{5}}{2}\right)^0 + D\left(\frac{1-\sqrt{5}}{2}\right)^0 = C \cdot 1 + D \cdot 1$
- $F_1 = C\left(\frac{1+\sqrt{5}}{2}\right)^1 + D\left(\frac{1-\sqrt{5}}{2}\right)^1 = C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right)$
- Solving for C and D, we get
- $C = \frac{1+\sqrt{5}}{2\sqrt{5}}$
- $D = \frac{-(1-\sqrt{5})}{2\sqrt{5}}$
- Substituting in, this yields
- $F_n = \frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-(1-\sqrt{5})}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$

Single-Root Case

- Our previous technique only works when the characteristic equation has distinct roots r and s
- Consider sequence $a_k = Aa_{k-1} + Ba_{k-2}$
- If $t^2 - At - B = 0$ only has a single root r, then the following sequences both satisfy a_k
  - $1, r^1, r^2, r^3, \ldots r^n, \ldots$
  - $0, r, 2r^2, 3r^3, \ldots nr^n, \ldots$

Single root equation

- Our old rule said that if $r_0, r_1, r_2, \ldots$ and $s_0, s_1, s_2, \ldots$ are sequences that satisfy the same SOLHRRwCC, then, for any $C, D \in \mathbb{R}$, the following sequence also satisfies the SOLHRRwCC
  - $a_n = Cr_n + Ds_n$, for integers $n \geq 0$
- For the single root case, this means that the explicit formula is:
  - $a_n = Cr^n + Dnr^n$, for integers $n \geq 0$
  - where C and D are determined by a_0 and a_1

SOLHRRwCC TL;DR

- To solve sequence $a_k = Aa_{k-1} + Ba_{k-2}$
- Find its characteristic equation $t^2 - At - B = 0$
- If the equation has two distinct roots r and s
  - Substitute a_0 and a_1 into $a_n = Cr^n + Ds^n$ to find C and D
- If the equation has a single root r
  - Substitute a_0 and a_1 into $a_n = Cr^n + Dnr^n$ to find C and D

General Recursive Definitions

Defining anything recursively

- We can define sets recursively
- To do so, we need three things:
  I. **Base:** A statement of that certain things are in the set
  II. **Recursion:** A set of rules saying how new objects can be shown to be in the set based on ones that are already known to be in the set
  III. **Restriction:** A statement that no objects belong to the set other than those coming from **I** and **II**

Example

- The set *P* of all strings of legal parenthesizations

I. **Base:** () is in *P*

II. **Recursion:**

   a. If *E* is in *P*, so is (*E*)

   b. If *E* and *F* are in *P*, so is *E F*

III. **Restriction:** No configurations of parentheses are in *P* other than those derived from **I** and **II**

Ackermann function

- Even functions can be defined recursively
- The Ackermann function is famous because it grows faster than any algebraic function
- It is defined for all non-negative integers as follows:
  - $A(0, n) = n + 1$
  - $A(m, 0) = A(m - 1, 1)$
  - $A(m, n) = A(m - 1, A(m, n - 1))$
- Find $A(1,2)$
- I won't make you find $A(4,4)$, because it is roughly $2^{2^{2^{65536}}}$

Set Theory

Sets

- A **set** is a collection of **elements**
- The set is defined entirely by its elements
  - Order doesn't matter
  - Repetitions don't matter (but, in general, we write each element of set only a single time)
- Examples
  - $\{1, 4, 9, 25\}$
  - $\{1\}$ is not the same as 1
  - $\{\ \}$ is the empty set

Notation

- For a finite set, we can write all of the elements inside curly braces
- For infinite sets, we don't have that luxury
- What's in each of the following sets?
  - $A = \{x \in \mathbb{Z} \mid -3 < x < 4\}$
  - $B = \{x \in \mathbb{Z}^+ \mid -3 < x < 4\}$
  - $C = \{x \in \mathbb{R} \mid -3 < x < 4\}$
  - $D = \{x \in \mathbb{Q} \mid -3 < x < 4\}$

Subsets

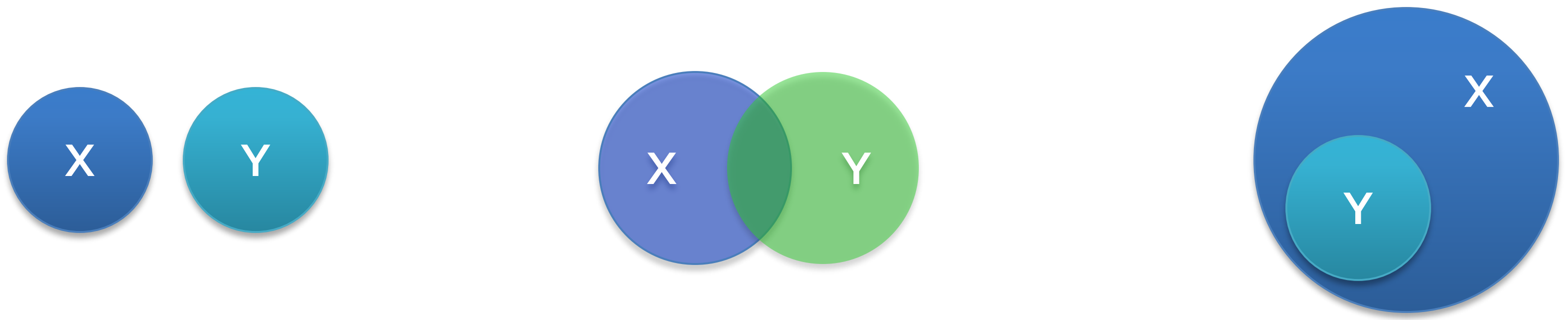
- Perhaps the simplest relationship between sets is the subset relationship
- We say that X is a subset of Y iff every element of X is an element of Y
- Notation $X \subseteq Y$
- Examples:
  - Is $\{ 1, 2, 3 \} \subseteq \{ 1, 2, 3, 4 \}$?
  - Is $\{ 1, 2, 3, 4 \} \subseteq \{ 1, 2, 3 \}$?
  - Is $\{ 1, 2, 3 \} \subseteq \{ 1, 2, 3 \}$?
  - Is $\{ \} \subseteq \{ 1, 2, 3 \}$?
  - Is $\{ \{1\}, \{2\}, \{3\} \} \subseteq \{ 1, 2, 3 \}$?

Proper subsets

- We say that X is a proper subset of Y iff every element of X is an element of Y, but there is some element of Y that is **not** an element of X
- Notation $X \subset Y$
- Examples:
  - Is $\{ 1, 2, 3 \} \subset \{ 1, 2, 3, 4 \}$?
  - Is $\{ 1, 2, 3, 4 \} \subset \{ 1, 2, 3 \}$?
  - Is $\{ 1, 2, 3 \} \subset \{ 1, 2, 3 \}$?
  - Is $\{ \} \subset \{ 1, 2, 3 \}$?
  - Is $\{ \{1\}, \{2\}, \{3\} \} \subset \{ 1, 2, 3 \}$?

Venn diagrams

- Venn diagrams show relationships between sets
- They can help build intuition about relationships, **but** they do not prove anything
- Also, their usefulness is limited to 2 or 3 sets
- Beyond that, they become difficult to interpret
- Example: For $X \not\subseteq Y$, there are 3 possibilities:



$\in$ vs. $\subseteq$

- The idea of being an element of a set is strongly tied to the idea of one set being the subset of another
- These two relationships are **different**, however
- $x \in X$ means that x is an element of X
- $Y \subseteq X$ means that Y is a subset of X
- Which of the following are true?
  - $2 \in \{1, 2, 3\}$
  - $\{2\} \in \{1, 2, 3\}$
  - $2 \subseteq \{1, 2, 3\}$
  - $\{2\} \subseteq \{1, 2, 3\}$
  - $\{2\} \subseteq \{\{1\}, \{2\}\}$
  - $\{2\} \in \{\{1\}, \{2\}\}$

Ticket Out the Door

Upcoming

Next time…

- Properties of sets
- Proofs with sets
- Russell's paradox and the Halting Problem

Reminders

- Finish Assignment 2
  - **Due tonight!**
- Assignment 3 is due next Friday
- Read 6.2, 6.3, and 6.4